

AN EXTENSION OF BAKHVALOV'S THEOREM FOR SYSTEMS OF CONSERVATION LAWS WITH DAMPING

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ABSTRACT. For 2×2 systems of conservation laws satisfying Bakhvalov conditions, we present a class of damping terms that still yield the existence of global solutions with periodic initial data of possibly large bounded total variation per period. We also address the question of the decay of the periodic solution. As applications we consider the systems of isentropic gas dynamics, with pressure obeying a γ -law, for the physical range $\gamma \geq 1$, and also for the “non-physical” range $0 < \gamma < 1$, both in the classical Lagrangian and Eulerian formulation, and in the relativistic setting. We give complete details for the case $\gamma = 1$, and also analyze the general case when $|\gamma - 1|$ is small. Further, our main result also establishes the decay of the periodic solution.

1. INTRODUCTION

In [13], Nishida shows that for the 2×2 system of compressible isothermic gas dynamics, where the pressure satisfies a γ -law with $\gamma = 1$, it is possible to construct a global solution using the Glimm method [8] for any initial data of bounded total variation, taking values in the physical region, with no smallness restriction on the value of the total variation. Following Nishida's work, Bakhvalov, in the important paper [1], establishes general conditions for a 2×2 system of conservation laws to enjoy the property of allowing the construction of a global solution by the Glimm method with no restriction of smallness in the value of the initial total variation. The purpose of this paper is to establish a general result extending the theorem of Bakhvalov, in [1], for 2×2 systems of conservation laws systems with damping. More specifically, we prescribe a general form for the damping term, with 4 degrees of freedom, and, besides Bakhvalov's conditions in [1], we impose additional conditions concerning our prescribed family of damping terms so as to obtain general conditions for the existence of a global solution, whose initial data may have large total variation, depending only on how big is the region where the original Bakhvalov's conditions are satisfied. Our main result also includes the decay of the periodic entropy solution of the system with damping. We then first apply the general result to the 2×2 system of isothermic gas dynamics, that is, pressure given by a so called γ -law with $\gamma = 1$, in Lagrangian and Eulerian coordinates, as well as

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the relativistic version. Finally, we also discuss the application of the main result to the general isentropic gas dynamics with pressure obeying a γ -law, when γ is sufficiently close to 1. In what follows we assume that the reader has some basic knowledge of the theory of conservation laws as presented, e.g., in [15, 14, 10, 3].

So, let us consider a 2×2 system of conservation laws in the form

$$(1.1) \quad \begin{cases} u_t + f_1(u, v)_x + g_1(u, v) = 0, \\ v_t + f_2(u, v)_x + g_2(u, v) = 0, \end{cases}$$

or in the form

$$(1.2) \quad \begin{cases} U_1(u, v)_t + f_1(u, v)_x + \tilde{g}_1(u, v) = 0, \\ U_2(u, v)_t + f_2(u, v)_x + \tilde{g}_2(u, v) = 0, \end{cases}$$

where $(u, v) \mapsto (U_1, U_2)$ is locally a change of coordinates, and the initial data

$$(1.3) \quad (u(x, 0), v(x, 0)) = (u_0(x), v_0(x)).$$

Let $\omega(U)$ and $\zeta(U)$ be a pair of Riemann invariants for (1.1) (or (1.2)), $U = (u, v)$, defined for $U \in \mathcal{V}$, with \mathcal{V} a domain in \mathbb{R}^2 . Let \bar{U} be given by $(\omega, \zeta)(\bar{U}) = (0, 0)$.

We recall that (1.1) (or (1.2)) is said to satisfy Bakhvalov's conditions (see [1]) on a domain \mathcal{V} if the following are satisfied, where $\zeta = L_i(\omega; \omega_r, \zeta_r)$ denotes i -left shock curve of the points that can be connected on the left by an i -shock to (ω_r, ζ_r) , and $\zeta = R_i(\omega; \omega_r, \zeta_r)$ denotes i -right shock curve of the points that can be connected on the right by an i -shock to (ω_r, ζ_r) , $i = 1, 2$:

- $B_1 :$ $\sup_{i, \mathcal{V}} |\lambda_i(\omega, \zeta)| < \infty$.
- $B_2 :$ $\forall (\omega, \zeta) \in \mathcal{V}, 0 < \frac{\partial R_1}{\partial \omega}, \frac{\partial L_1}{\partial \omega} < 1, 1 < \frac{\partial R_2}{\partial \omega}, \frac{\partial L_2}{\partial \omega} < +\infty, w \neq w_0$
- $B_3 :$ If $\zeta_r = R_i(\omega_r; \omega_l, \zeta_l)$, $i = 1, 2$, then shock curves $\zeta = R_i(\omega; \omega_l, \zeta_l)$, $\omega \leq \omega_l$ and $\zeta = L_i(\omega; \omega_r, \zeta_r)$, $\omega \geq \omega_r$ intersect only in points (ω_l, ζ_l) , (ω_r, ζ_r) .
- $B_4 :$ If four points (ω_l, ζ_l) , (ω_r, ζ_r) , (ω_m, ζ_m) and $(\hat{\omega}_m, \hat{\zeta}_m)$ satisfy $\zeta_m = R_2(\omega_m; \omega_l, \zeta_l)$, $\zeta_r = R_1(\omega_r; \omega_m, \zeta_m)$, $\hat{\zeta}_m = R_1(\hat{\omega}_m; \omega_l, \zeta_l)$ and $\zeta_r = R_2(\omega_r; \hat{\omega}_m, \hat{\zeta}_m)$, then $(\omega_l - \hat{\omega}_m) + (\hat{\zeta}_m - \zeta_r) \leq (\zeta_l - \zeta_m) + (\omega_m - \omega_r)$.

We make the following assumptions:

- (D1)** We assume that (1.1) (or (1.2)) is a strictly hyperbolic genuinely nonlinear system and that, with respect to the pair (ω, ζ) , (1.1) (or (1.2)) satisfies the Bakhvalov conditions B_1 – B_4 on \mathcal{V} . These, in particular, ensure that both ω and ζ decrease across the shocks of both families on \mathcal{V} , and, if $\mathfrak{R}(x, t; U_L, U_M)$ and $\mathfrak{R}(x, t; U_M, U_R)$ are the solutions of the Riemann problems formed for the two pairs of constant states (U_L, U_M) and (U_M, U_R) ,

respectively, and $\mathfrak{R}(x, t; U_L, U_R)$ is the solution of the Riemann problem for the pair (U_L, U_R) , then

$$(1.4) \quad \mathcal{L}(\mathfrak{R}(U_L, U_R)) \leq \mathcal{L}(\mathfrak{R}(U_L, U_M)) + \mathcal{L}(\mathfrak{R}(U_M, U_R)),$$

where

$$(1.5) \quad \mathcal{L}(\mathfrak{R}(x, t; U_L, U_R)) := \sum_{1\text{-shocks}} |[\omega(\mathfrak{R})]| + \sum_{2\text{-shocks}} |[\zeta(\mathfrak{R})]|,$$

and $[\omega(\mathfrak{R})] = \omega(U_{\text{left}}) - \omega(U_{\text{right}})$, U_{left} being the state to the left of the shock wave, and U_{right} the one to the right.

(D2) In the case of system (1.1), we assume that $G(U) = (g_1(U), g_2(U))$, satisfies

$$(1.6) \quad G(U) = a\omega \mathbf{r}_1 + b\zeta \mathbf{r}_2,$$

where $a > 0, b > 0$ and $\mathbf{r}_1, \mathbf{r}_2$ are normalized right-eigenvectors such that

$$(1.7) \quad \mathbf{r}_1 \cdot \nabla \omega = 1, \quad \mathbf{r}_1 \cdot \nabla \zeta = 0, \quad \mathbf{r}_2 \cdot \nabla \omega = 0, \quad \mathbf{r}_2 \cdot \nabla \zeta = 1.$$

In the case of system (1.2), denoting

$$(1.8) \quad A(u, v) = \begin{bmatrix} \frac{\partial U_1}{\partial u} & \frac{\partial U_1}{\partial v} \\ \frac{\partial U_2}{\partial u} & \frac{\partial U_2}{\partial v} \end{bmatrix},$$

we then have that $\tilde{G}(u, v) = (\tilde{g}_1(u, v), \tilde{g}_2(u, v))$ is defined by the formula

$$(1.9) \quad \tilde{G}(u, v) := A(u, v)G(u, v),$$

where $G(u, v)$ is given by (1.6) and now $\mathbf{r}_1, \mathbf{r}_2, \omega, \zeta$ are eigenvectors and Riemann invariants for the matrix $A^{-1}(U)\nabla f(U)$, with $f = (f_1, f_2)$.

(D3) There exists $\delta_0 > 0$, such that, for $0 < \delta < \delta_0$, if U_L, U_R are connected by a 1-shock wave (or a 2-shock wave), $\Lambda := \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, with $a > 0, b > 0$, as in the previous item, the Riemann solution (in the plane of Riemann invariants) with left state $\tilde{Z}_L := e^{-\Lambda\delta} Z_L$ and right state $\tilde{Z}_R := e^{-\Lambda\delta} Z_R$, with $Z_L := (\omega_L, \zeta_L)$, $Z_R := (\omega_R, \zeta_R)$, $\omega_L = \omega(U_L)$, etc., satisfies

$$(1.10) \quad \mathcal{L}(\mathfrak{R}(\tilde{Z}_L, \tilde{Z}_R)) \leq \mathcal{L}(\mathfrak{R}(Z_L, Z_R)) = \omega(U_L) - \omega(U_R) \\ (= \zeta(U_L) - \zeta(U_R), \text{ in the case of a 2-shock}).$$

(D4) The system (1.1) admits a strictly convex entropy η_* satisfying $\eta_*(\bar{U}) = 0$, $\nabla \eta_*(\bar{U}) = 0$, and

$$(1.11) \quad \nabla_U \eta_*(U) \cdot G(U) = a\omega \partial_\omega \eta_*(U(\omega, \zeta)) + b\zeta \partial_\zeta \eta_*(U(\omega, \zeta)) \geq 0,$$

which, when $a = b$, simply means that η_* is radially nondecreasing in the Riemann invariants (ω, ζ) -plane. We recall that this means that there exists a companion function q_* , the entropy flux, such that $\nabla \eta_* \nabla f = \nabla q_*$.

We remark that, from **(D2)**, we see, in particular, that our family of damping terms, given by (1.6) (or (1.9)) has 4 degrees of freedom: 2 corresponding to the choice of \bar{U} , and 2 corresponding to the choice of a and b in (1.6).

We assume that (u_0, v_0) is a periodic function, with period, say, 1, with local bounded total variation, and we set $\beta_0 := \text{TV}_{\text{per}}(u_0, v_0)$, where TV_{per} stands for total variation in x over one period. Let

$$\mu_0 := \int_0^1 \eta_*(U_0(x)) dx,$$

$$\Sigma_0 = \{U \in \mathbb{R}^2 : \eta_*(U) \leq \mu_0\},$$

and

$$\Sigma_{0,R} := \{U \in \mathbb{R}^2 : \text{dist}(U; \Sigma_0) < R\}.$$

Let $\mathcal{V}_0 \subseteq \mathbb{R}^2$ be the invariant domain

$$\mathcal{V}_0 := \{U \in \mathbb{R}^2 : \omega(U) \leq \sup_{x \in [0,1]} \omega(U_0(x)), \zeta(U) \geq \inf_{x \in [0,1]} \zeta(U_0(x))\}.$$

Theorem 1.1. *There exists a constant $C_0 > 0$, depending only on the flux functions in (1.1) (or (1.2)), such that if*

$$\Sigma_{0,C_0\beta_0} \cap \mathcal{V}_0 \subseteq \mathcal{V},$$

the problem (1.1),(1.3) (or (1.2),(1.3)) possesses a global entropy solution $U(x, t)$, with locally bounded total variation, such that

$$(1.12) \quad U(x, t) \in \Sigma_{0,C_0\beta_0} \cap \mathcal{V}_0, \quad \text{for all } (x, t) \in \mathbb{R} \times [0, \infty)$$

Moreover, if η_ is strictly dissipative in the sense that, except at $U = \bar{U}$, the strict inequality holds in (1.11), we have*

$$(1.13) \quad \lim_{t \rightarrow \infty} \int_0^1 |U(x, t) - \bar{U}| dx = 0.$$

The proof of Theorem 1.1 is given in Section 2. In Section 3, we apply Theorem 1.1 to the 2×2 system of compressible esoteric gas dynamics, in which the pressure is given by a γ -law with $\gamma = 1$. Finally, in Section 4, we address the application of Theorem 1.1 to the case of a general γ -law pressure with γ sufficiently close to 1.

2. PROOF OF THEOREM 1.1

In this section we prove Theorem 1.1.

We construct an approximate solution for (1.1),(1.3) by using a fractional step procedure where, for each time interval $[nh, (n+1)h]$, we make two successive iterations in the following way. Assume the approximate solution has been defined at $t = nh$, such that $U^h(x, nh) = U^h((m + a_n)l, nh)$, for $x \in ((m - \frac{1}{2})l, (m + \frac{1}{2})l)$,

for some randomly chosen number $a_n \in (-\frac{1}{2}, \frac{1}{2})$, where $m \in \mathbb{Z}$, and $l = 1/N_l$, for some $N_l \in \mathbb{N}$, satisfies the usual CFL-condition

$$\frac{l}{h} \geq \sup_{U \in \mathcal{V}} \{|\lambda_i(U)| : i = 1, 2\}.$$

As a first step, we define $\hat{U}^h(x, t)$, for $(x, t) \in \mathbb{R} \times [nh, (n+1)h)$, according to Glimm's method, that is, defining it over the mesh rectangle $((m - \frac{1}{2})l, (m + \frac{1}{2})l) \times [nh, (n+1)h)$ as the Riemann solution centered at (ml, nh) with left state $U^h((m + a_n)l, nh)$ and right state $U^h((m + 1 + a_n)l, nh)$. In the second step, we define $U^h(x, t)$, for $nh \leq t < (n+1)h$, $x \in \mathbb{R}$, as the solution of

$$(2.1) \quad \begin{cases} \frac{dU}{dt}(x, t) = -G(U(x, t)), & nh < t < (n+1)h \\ U(x, nh) = \hat{U}^h((m + a_{n+1})l, (n+1)h - 0), & \text{for } (m - \frac{1}{2})l < x < (m + \frac{1}{2})l, \end{cases}$$

again for some randomly chosen number $a_{n+1} \in (-\frac{1}{2}, \frac{1}{2})$. We close the recursive definition of $U^h(x, t)$ by setting

$$U^h(x, (n+1)h) = U^h((m + a_{n+1})l, (n+1)h - 0), \quad \text{for } (m - \frac{1}{2})l < x < (m + \frac{1}{2})l.$$

Let us also define

$$\hat{U}(x, (n+1)h) = \hat{U}^h((m + a_{n+1})l, (n+1)h - 0), \quad \text{for } (m - \frac{1}{2})l < x < (m + \frac{1}{2})l,$$

which would be the prescription for constructing the Glimm solution, for the corresponding homogeneous system.

From the above construction, we see that, in each time interval $t \in [nh, (n+1)h)$, $U^h(x, t)$ is piecewise constant in x , for each fixed t . We define the functional

$$(2.2) \quad \mathcal{L}(U^h(x, nh)) := \sum_{1\text{-shocks}}^{\text{per}} [\omega(U^h)] + \sum_{2\text{-shocks}}^{\text{per}} [\zeta(U^h)],$$

where $\sum_{1\text{-shocks}}^{\text{per}}$ means that the sum is over all 1-shocks over one period resulting from the solution of the Riemann problems for each of the discontinuities over one period and $[\omega(U)] = \omega(U_{\text{left}}) - \omega(U_{\text{right}})$, with a similar definition for $\sum_{2\text{-shocks}}^{\text{per}}$ and $[\zeta(U)]$.

We recall that Bakhvalov's theorem implies that

$$(2.3) \quad \mathcal{L}(\hat{U}^h(x, (n+1)h)) \leq \mathcal{L}(U^h(x, nh)).$$

To get the desired inequality

$$(2.4) \quad \mathcal{L}(U^h(x, (n+1)h)) \leq \mathcal{L}(U^h(x, nh)),$$

it then suffices to prove

$$(2.5) \quad \mathcal{L}(U^h(x, (n+1)h)) \leq \mathcal{L}(\hat{U}^h(x, (n+1)h)).$$

Observe that, from (2.1), in passing from \hat{U}^h to U^h , the Riemann invariants satisfy

$$(2.6) \quad \begin{aligned} \frac{d\omega}{dt} &= -a\omega, \\ \frac{d\zeta}{dt} &= -b\zeta. \end{aligned}$$

To obtain inequality (2.5), we first observe that condition **(D3)** implies that inequality (1.10) also holds if U_L and U_4 are any two constant states, not necessarily connected by a 1-shock or a 2-shock, as a consequence of Bakhvalov condition, where we use the fact that if P and Q are connected by a rarefaction wave of the first or the second family, then $e^{-\Lambda\delta}P$ and $e^{-\Lambda\delta}Q$ are also connected by a rarefaction wave of the same family. Therefore, inequality (2.5) follows from the validity of the corresponding inequality for each individual discontinuity and its corresponding transformation by the application of $e^{-\Lambda h}$ to both points forming the discontinuity, written in the Riemann invariants coordinates. This suffices to prove the uniform boundedness of the total variation per period for $U^h(x, t)$.

We now briefly explain how the control of the L^∞ norm of U^h is achieved. For that we need to use the strictly convex entropy η_* whose existence is assumed. Because of condition **(D4)**, we have

$$(2.7) \quad \begin{aligned} \eta_*\left(\int_0^1 U^h(x, t) dx\right) &\leq \int_0^1 \eta_*(U^h(x, t)) dx \leq \int_0^1 \eta_*(U_0^h(x)) dx \\ &+ \sum_{j=1}^{j=[t/h]} \int_0^1 (\eta_*(U^h(x, jh - 0)) - \eta_*(U^h(x, jh + 0))) dx. \end{aligned}$$

Observe that, as proven in [11], the summation at the end of (2.7) is nonpositive in the limit as $h \rightarrow 0$ and it is always bounded by $C\beta_0 t$, for some constant $C > 0$. Since η_* is strictly convex, this means that the mean value $\int_0^1 U^h(x, t) dx$ is always inside the region Σ_0 , in a sufficiently small time interval. Since, as we have just proven, the total variation per period is bounded at any time by $\frac{1}{2}C_0\beta_0$, for some constant C_0 only depending on the system, we conclude that $U^h(x, t)$ assumes its values in $\Sigma_{0, C_0\beta_0}$. This argument should be performed in a stepwise manner in intervals of time of a fixed length T , extracting subsequences for an appropriate choice of the random sequence $\{a_n\}$ by means of a diagonal argument, as in [6, 7].

Now, the proof of the consistence of the above Glimm scheme with fractional step follows by standards arguments whose central point relies on the original consistence argument in [8], which then implies the convergence of U^h to a bounded function with bounded total variation per period U which is a weak solution of (1.2) (or (1.2), as the case may be). Also, by the same argument in [11], we obtain that U satisfies the entropy inequality

$$(2.8) \quad \int_{\mathbb{R} \times (0, \infty)} \{\eta_*(U)\varphi_t + q_*(U)\varphi_x\} dx dt + \int_{\mathbb{R}} \eta_*(U_0(x))\varphi(x, 0) dx \geq 0,$$

for any $0 \leq \varphi \in C_0^\infty(\mathbb{R}^2)$.

To prove the decay property (1.12) we follow the approach in [2] based on the compactness of the scaling sequence $U^\varepsilon(x, t) = U(\varepsilon^{-1}x, \varepsilon^{-1}t)$ as $\varepsilon \rightarrow 0$. We notice that U^ε is an entropy weak solution of a system like (1.1) (or (1.2)) only that instead of $G(U)$ (resp., $\tilde{G}(U)$), we now have $\varepsilon^{-1}G(U^\varepsilon)$ (resp., $\varepsilon^{-1}\tilde{G}(U^\varepsilon)$). Clearly, inequality (2.8) is also satisfied by U^ε and this implies, through a standard argument in the theory of distributions, that

$$\eta_*(U^\varepsilon)_t + q_*(U^\varepsilon)_x \in \{\text{bounded subset of } \mathcal{M}_{\text{loc}}^-(\mathbb{R}^2 \times (0, \infty))\},$$

where $\mathcal{M}_{\text{loc}}^-(\mathbb{R}^2 \times (0, \infty))$ denotes the space of nonpositive Radon measures of locally bounded total variation. Now, under the assumption that (η_*, q_*) is strictly dissipative, this implies that for any entropy-entropy flux pair (η, q) for the system (1.1) (or (1.2)), satisfying $\eta(\bar{U}) = 0$, $\nabla \eta(\bar{U}) = 0$, we have

$$\eta(U^\varepsilon)_t + q(U^\varepsilon)_x \in \{\text{bounded subset of } \mathcal{M}_{\text{loc}}(\mathbb{R}^2 \times (0, \infty))\},$$

where $\mathcal{M}_{\text{loc}}(\mathbb{R}^2 \times (0, \infty))$ denotes the space of Radon measures of locally bounded total variation. Indeed, this follows from the fact that, under the strictly dissipative condition, given any such pair (η, q) we may find a constant C_η such that $\eta + C_\eta \eta_*$ satisfies the dissipative condition (1.11) on a given compact domain. Here we also use the strict convexity of η_* to handle the dissipative inequality around \bar{U} . Therefore, through a well know interpolation result (see, *e.g.*, [12, 17]) it follows that

$$\eta(U^\varepsilon)_t + q(U^\varepsilon)_x \in \{\text{compact subset of } H_{\text{loc}}^{-1}(\mathbb{R}^2 \times (0, \infty))\}.$$

This allows us to apply the DiPerna's compactness theorem for general 2×2 strictly hyperbolic genuinely nonlinear systems in [5]. Hence, we deduce that U^ε is compact in $L_{\text{loc}}^1(\mathbb{R} \times (0, 1))$. Now, taking any convergent subsequence, still denoted by U^ε , we see that, if $U^\varepsilon \rightarrow \tilde{U}$ in $L_{\text{loc}}^1(\mathbb{R} \times (0, \infty))$, we must have

$$G(\tilde{U}(x, t)) = 0, \quad (\text{or } \tilde{G}(\tilde{U}(x, t)), \text{ in the case of (1.2)}),$$

for a.e. $(x, t) \in \mathbb{R} \times (0, \infty)$, which implies that $\tilde{U}(x, t) \equiv \bar{U}$, a.e. In particular, we deduce that the whole sequence U^ε converges to \bar{U} , a.e. Now using the same argument as in [2] we finally conclude that (1.13) holds.

3. APPLICATION TO ISOTHERMIC GAS DYNAMICS

In this section we present, as an application of Theorem 1.1, the system of isothermic gas dynamics, first in classical mechanics, both in Lagrangian and Eulerian coordinates, and also in relativistic mechanics.

We consider first the case where the corresponding homogeneous system (i.e., without damping) is the p -system of isothermic gas dynamics

$$(3.1) \quad \begin{cases} u_t - v_x = 0, \\ v_t + p(u)_x = 0. \end{cases}$$

Here, u represents the specific volume, v represents the velocity, and the pressure is given by $p(u) = \kappa u^{-\gamma}$, with $\kappa > 0$, which is the so called γ -law for the pressure. Here we will consider the case of an isothermic gas which corresponds to the adiabatic exponent $\gamma = 1$. For simplicity we take $\kappa = 1$. We take $\mathcal{V} = \{(u, v) \in \mathbb{R}^2 : u > 0\}$. In this case, for $\bar{U} = (\bar{u}, \bar{v})$, with $\bar{u} > 0$, the Riemann invariant are the classical ones

$$(3.2) \quad \omega = v - \bar{v} + \log \frac{u}{\bar{u}}, \quad \zeta = v - \bar{v} - \log \frac{u}{\bar{u}},$$

Therefore, in this case, we have

$$(3.3) \quad \mathbf{r}_1 = \frac{1}{2}(u, 1), \quad \mathbf{r}_2 = \frac{1}{2}(-u, 1).$$

Hence, if $a = b$, G has the form

$$\begin{aligned} G(U) &= \frac{a}{2}((v - \bar{v} + \log \frac{u}{\bar{u}})(u, 1) + (v - \bar{v} - \log \frac{u}{\bar{u}})(-u, 1)) \\ &= a(u \log \frac{u}{\bar{u}}, v - \bar{v}). \end{aligned}$$

Finally, we easily verify that the well known convex entropy

$$\eta(u, v) = \frac{1}{2}(v - \bar{v})^2 - \log \frac{u}{\bar{u}} + \frac{1}{\bar{u}}(u - \bar{u}),$$

with entropy flux $q(u, v) = (v - \bar{v})(1/u - 1/\bar{u})$, satisfies condition **(D4)** and it is clearly strictly dissipative in the case $a = b$. Actually, when $a = b$, the strict dissipation follows from the strict convexity since in this case (1.11) amounts to require that η is strictly increasing with respect to the radius in a polar coordinates system centered at \bar{U} . Thus, for $\gamma = 1$ and $a = b$, (3.1) with damping becomes

$$(3.4) \quad \begin{cases} u_t - v_x = a u \log \frac{u}{\bar{u}}, \\ v_t + (\frac{1}{u})_x = a(v - \bar{v}). \end{cases}$$

System (3.1) is also frequently presented in the so called Eulerian coordinates, where it takes a form like (1.2). In this case, we get the following system

$$(3.5) \quad \begin{cases} \rho_t + (\rho v)_x = 0, \\ (\rho v)_t + (\rho v^2 + p(\rho))_x = 0. \end{cases}$$

Here, ρ represents the density, v is again the velocity, and the pressure is given by $p(\rho) = \kappa \rho^\gamma$. Again we consider the esoteric case where $\gamma = 1$, and we take $\kappa = 1$.

We take $\mathcal{V} = \{(\rho, v) \in \mathbb{R}^2 : \rho > 0\}$. Now, with $\bar{U} = (\bar{\rho}, \bar{v})$, the expressions for the classical Riemann invariants take the forms

$$(3.6) \quad \omega = v - \bar{v} + \log \frac{\rho}{\bar{\rho}}, \quad \zeta = v - \bar{v} - \log \frac{\rho}{\bar{\rho}},$$

and the right-eigenvectors are given by

$$(3.7) \quad \mathbf{r}_1 = \frac{1}{2}(\rho, 1), \quad \mathbf{r}_2 = \frac{1}{2}(-\rho, 1).$$

Therefore, if $a = b$,

$$G(\rho, v) = a(\rho \log \frac{\rho}{\bar{\rho}}, v - \bar{v}).$$

Now, we have $U_1(\rho, v) = \rho$, $U_2(\rho, v) = \rho v$, and so

$$A(\rho, v) = \begin{bmatrix} 1 & 0 \\ v & \rho \end{bmatrix}.$$

In this case, a simple calculation shows that

$$\tilde{G}(\rho, v) = a(\rho \log \frac{\rho}{\bar{\rho}}, \rho(v - \bar{v}) + \rho v \log \frac{\rho}{\bar{\rho}}).$$

Also, concerning condition **(D4)**, the system (3.5) admits the following well known convex entropy

$$\eta(\rho, v) = \frac{1}{2}\rho(v - \bar{v})^2 + \rho \log \frac{\rho}{\bar{\rho}} - \rho + \bar{\rho},$$

which is clearly strictly dissipative in the case $a = b$, for the already mentioned reason. Thus, for $a = b$, (3.5) with damping becomes

$$(3.8) \quad \begin{cases} \rho_t + (\rho v)_x = a \rho \log \frac{\rho}{\bar{\rho}}, \\ (\rho v)_t + (\rho v^2 + p(\rho))_x = a(\rho(v - \bar{v}) + \rho v \log \frac{\rho}{\bar{\rho}}). \end{cases}$$

Another example of the importance of having the prescription for the damping for systems in the more general form (1.2) is provided by the relativistic version of the Euler equations of isentropic gas dynamics, namely,

$$(3.9) \quad \begin{cases} \frac{\partial}{\partial t} U_1(\rho, v) + \frac{\partial}{\partial x} U_2(\rho, v) = 0, \\ \frac{\partial}{\partial t} U_2(\rho, v) + \frac{\partial}{\partial x} (U_2(\rho, v)v + p(\rho)) = 0, \end{cases}$$

where

$$(3.10) \quad \begin{aligned} U_1 &= \frac{v}{c^2} U_2 + \rho \\ U_2 &= (p + \rho c^2) \frac{v}{c^2 - v^2}, \\ p(\rho) &= \kappa^2 \rho^\gamma, \end{aligned}$$

where $\gamma \geq 1$. Here we just consider the case $\gamma = 1$; κ is a positive constant, and c is the speed of light. Also, for simplicity we only write the formulas for $\bar{U} = (1, 0)$. Here we take $\mathcal{V} = \{(\rho, v) \in \mathbb{R}^2 : \rho > 0, |v| \leq c\}$. Existence of a global solution in BV of (3.9), with $\gamma = 1$, for initial data in BV , with no smallness restriction on the total variation, was obtained in [16], where it was observed that the shock

curves for this system enjoy the same features as those for (3.1) and (3.5), which allowed Nishida's global existence result for large initial total variation.

In this case, we have

$$A(\rho, v) = \begin{bmatrix} (\kappa^2 + c^2) \frac{v^2}{c^2(c^2 - v^2)} + 1 & (\kappa^2 + c^2) \rho \frac{2v}{(c^2 - v^2)^2} \\ (\kappa^2 + c^2) \frac{v}{c^2 - v^2} & (\kappa^2 + c^2) \rho \frac{c^2 + v^2}{(c^2 - v^2)^2} \end{bmatrix}.$$

On the other hand, the classical Riemann invariants are given by the expressions

$$(3.11) \quad \begin{aligned} \omega &:= \frac{c}{2} \log \frac{c+v}{c-v} + \frac{c^2}{\kappa^2 + c^2} \log \rho, \\ \zeta &:= \frac{c}{2} \log \frac{c+v}{c-v} - \frac{c^2}{\kappa^2 + c^2} \log \rho. \end{aligned}$$

The Riemann invariants for any \bar{U} are $\omega(U) - \omega(\bar{U})$, $\zeta(U) - \zeta(\bar{U})$. For the right-eigenvalues we then have the expressions

$$(3.12) \quad \begin{aligned} \mathbf{r}_1 &:= \frac{1}{2} \left(\rho \left(1 + \frac{\kappa^2}{c^2} \right), 1 - \frac{v^2}{c^2} \right) \\ \mathbf{r}_2 &:= \frac{1}{2} \left(-\rho \left(1 + \frac{\kappa^2}{c^2} \right), 1 - \frac{v^2}{c^2} \right). \end{aligned}$$

Using the formula

$$\tilde{G}(\rho, v) := A(\rho, v)G(\rho, v),$$

with

$$G(\rho, v) := a(\zeta \mathbf{r}_1 + \omega \mathbf{r}_2),$$

we obtain the prescribed formula for the damping in the relativistic isentropic Euler equations of gas dynamics

$$(3.13) \quad \begin{cases} \frac{\partial}{\partial t} U_1(\rho, v) + \frac{\partial}{\partial x} U_2(\rho, v) + \tilde{g}_1(\rho, v) = 0, \\ \frac{\partial}{\partial t} U_2(\rho, v) + \frac{\partial}{\partial x} (U_2(\rho, v)v + p(\rho)) + \tilde{g}_2(\rho, v) = 0, \end{cases}$$

To obtain a convex entropy for the system (3.9) we may just apply the general result of Lax in [11], which establishes the existence of such a convex entropy defined on any compact over which the system is strictly hyperbolic and genuinely nonlinear. The strict convexity implies the strictly dissipative property in condition **(D4)**.

Condition **(D1)**, that is, the fact that the system satisfies Bakhvalov's conditions, in all the above formulations for the isothermic gas dynamics system, is well known. The key fact to be used here is that the shock curves for these systems, in the Riemann invariants plane, belonging to each of the two characteristic families, are all translations of the same curve based on a fixed state in the Riemann invariants plane. Condition **(D2)**, in all these examples, is satisfied since the damping terms satisfy the prescribed formula in this condition.

As for condition **(D3)**, using the property enjoyed by the shock curves just mentioned, the inequality (1.4) can be proved with the help of Fig.1, in the following

manner. In Fig. 1 we see a 1-shock from P_1 to P_2 , and the Riemann solution for the discontinuity between $P'_1 = e^{-ah} P_1$ and $P'_2 = e^{-ah} P_2$. These are the images of P_1 and P_2 when we pass from \hat{U}^h to U^h , but to get a better view of what is going on and be able to compare lengths we translate the entire composite wave curve representing the Riemann solution so that P'_1 assumes the position of P_1 . Due to the fact that 1-shock curves are translations of one another, after the mentioned translation, have the coinciding two 1-shock curves as shown in Fig. 1. By the second Bakhvalov condition, the slope of the shock curves is greater than 0 and less than 1. We then deduce that

$$\begin{aligned}\omega(P_1) - \omega(P_2) &= \omega(P_1) - \omega(P'_3) + \omega(P'_3) - \omega(P_2) \\ &\geq \omega(P_1) - \omega(P'_3) + \zeta(P'_3) - \zeta(P_2) \\ &\geq \omega(P_1) - \omega(P'_3) + \zeta(P'_3) - \zeta(P'_2),\end{aligned}$$

where for the inequality in the second line we use the fact that for a 1-shock curve $\Delta\omega \geq \Delta\zeta$, by Bakhvalov condition B_2 , and the inequality in the third line is obvious. We also observe, concerning the inequality in the second line, that we are taking advantage from the fact, peculiar for the γ -law case $\gamma = 1$, that the shock curves are translations from one another. Similarly, we prove the corresponding inequality for the 2-shock waves (see Fig.2). In this case, after translating the composite wave curve representing the Riemann solution, to improve our way to compare lengths, we further translate the 2-shock curve in the translated Riemann solution, so as to make it lie on the original 2-shock curve connecting P_1 and P_2 , and P'_2 goes over to P''_3 , and from P''_3 to P'_2 we draw the translated 1-shock, which originally connects P_1 and the middle state P'_3 . In particular, we have $\zeta(P'_3) - \zeta(P'_2) = \zeta(P_1) - \zeta(P''_3)$ and $\omega(P_1) - \omega(P'_3) = \omega(P''_3) - \omega(P'_2)$, and so, using the fact that in a 2-shock we have $\Delta\zeta \geq \Delta\omega$, we get

$$\begin{aligned}\zeta(P_1) - \zeta(P_2) &= \zeta(P_1) - \zeta(P''_3) + \zeta(P''_3) - \zeta(P_2) \\ &\geq \zeta(P_1) - \zeta(P''_3) + \omega(P''_3) - \omega(P_2) \\ &\geq \zeta(P_1) - \zeta(P''_3) + \omega(P''_3) - \omega(P'_2) \\ &= \omega(P_1) - \omega(P'_3) + \zeta(P'_3) - \zeta(P'_2).\end{aligned}$$

These facts imply that inequality (1.4) holds, as was to be proved.

4. APPLICATION TO THE GENERAL γ -LAW FOR γ CLOSE TO 1

In this section we outline the application of Theorem 1.1 to the general γ -law compressible isentropic gas dynamics of γ sufficiently close to 1.

First we recall that in [1] it was proven that systems (3.1) and (3.5) with $0 < \gamma < 1$ satisfy Bakhvalov's conditions B_1 – B_4 , for the usual Riemann invariants, namely,

$$(4.1) \quad \begin{aligned} \omega &= v - \bar{v} - \frac{2\gamma^{1/2}\kappa}{\gamma-1}(u^{-(\gamma-1)/2} - \bar{u}^{-(\gamma-1)/2}), \\ \zeta &= v - \bar{v} + \frac{2\gamma^{1/2}\kappa}{\gamma-1}(u^{-(\gamma-1)/2} - \bar{u}^{-(\gamma-1)/2}), \end{aligned}$$

for (3.1), while the corresponding Riemann invariants for (3.5) are obtained by just making $u = 1/\rho$, $\bar{u} = 1/\bar{\rho}$ in (4.1).

As observed in [7], the same is true for the relativistic version (3.9), for $p(\rho) = \kappa^2 \rho^\gamma$. Namely, when $0 < \gamma < 1$, Bakhvalov's conditions B_1 – B_4 are satisfied for the standard Riemann invariants given by (cf. [16])

$$(4.2) \quad \begin{aligned} \omega &= \frac{1}{2} \log \frac{c+v}{c-v} - \frac{1}{2} \log \frac{c+\bar{v}}{c-\bar{v}} + c \int_{\bar{\rho}}^{\rho} \frac{\sqrt{p'(s)}}{p(s) + sc^2} ds \\ \zeta &= \frac{1}{2} \log \frac{c+v}{c-v} - \frac{1}{2} \log \frac{c+\bar{v}}{c-\bar{v}} - c \int_{\bar{\rho}}^{\rho} \frac{\sqrt{p'(s)}}{p(s) + sc^2} ds. \end{aligned}$$

On the other hand, for $\gamma > 1$, DiPerna [4] introduced a new family of Riemann invariants for (3.1) and (3.5) for which these systems satisfy Bakhvalov's conditions B_1 – B_4 over certain region contained in the physical domain $\{\rho > 0\}$ in the Riemann invariants plane, which includes any compact subset of the physical domain, as long as $\gamma > 1$ is sufficiently close to 1. DiPerna's Riemann invariants, when properly normalized, converge locally uniformly with all their derivatives to the standard Riemann invariants for $\gamma = 1$ (cf. (3.2), (3.6)) as $\gamma \rightarrow 1+$.

In [7] this fact was extended to the relativistic version (3.9). Namely, in [7] it is proven that DiPerna's formula defining the family of new Riemann invariants in terms of the classical Riemann invariants provide also in the relativistic case a pair of Riemann invariants for which Bakhvalov's conditions B_1 – B_4 are satisfied in a region \mathcal{V} in the physical domain $\{\rho > 0, v^2 < c^2\}$ which includes any compact in the physical domain, as long as $\gamma > 1$ is sufficiently close to 1.

So, concerning condition **(D1)**, for $0 < \gamma < 1$, for systems (3.1) or (3.5), we can take the whole Riemann invariants plane as \mathcal{V} , while for the relativistic version (3.9) we may take as \mathcal{V} the subset of the Riemann invariants plane corresponding to physical domain $\{v^2 < c^2\}$. On the other hand, for $1 < \gamma < 2$, for systems (3.1) or (3.5), we may take as \mathcal{V} the region of the Riemann invariants plane for which DiPerna's Riemann invariants satisfy Bakhvalov's conditions B_1 – B_4 , which includes any compact contained in the physical domain $\{\rho > 0, v^2 < c^2\}$, as long as $\gamma > 1$ is sufficiently close to 1.

Condition **(D2)** only prescribe the formula for the damping terms. As for **(D4)**, it is satisfied in the general case, $0 < \gamma$, when $a = b$.

Therefore, it only remains to discuss condition **(D3)** for $\beta(\gamma) := |\gamma - 1| > 0$. We keep assuming $a = b$. We claim that condition **(D2)** is also satisfied in any

compact region in the physical domain, for $\beta(\gamma) > 0$ sufficiently small. So, let K be a given compact region in the physical domain in the Riemann invariants plane. Let also $V \supset K$ be a neighborhood of K with compact closure contained in the physical domain and $\partial V \in C^\infty$. We introduce the functions $\Theta_1(\gamma, \delta, W_1, W_2)$ and $\Theta_2(\gamma, \delta, Z_1, Z_2)$, with $W_1, Z_1 \in K$, $W_2, Z_2 \in \bar{V}$, W_2 is connected to the right to W_1 by a 1-shock curve relatively to the γ -law system, in either classical or relativistic version, Z_2 is connected to the right to Z_1 by a 2-shock curve also pertaining to the γ -law system in either case, and $0 < \delta \leq 1$. Θ_1 is defined as follows. Given $\gamma > 0$, $0 < \delta \leq 1$, and $W_1 \in K$, for each W_2 in the right 1-shock curve pertaining to the γ -law, say, in the classical version, we consider the Riemann problem between $W'_1 = e^{-a\delta}W_1$, as a left state, and $W'_2 = e^{-a\delta}W_2$, as a right state. It is known that this Riemann problem is solved by drawing a 1-shock curve issuing from W'_1 and an inverse 2-wave curve issuing from W'_2 , consisting of the states that can be connected to the left to W'_2 by either a 2-shock or a 2-rarefaction wave. Then, we translate the composite wave curve representing the Riemann solution together with the line connecting W'_1 to W'_2 , which is parallel to that connecting W_1 and W_2 , until the line connecting W'_1 to W'_2 lies entirely over the line connecting W_1 to W_2 , with W'_1 coinciding with W_1 . Denote by W'_3 the translated middle state in the Riemann solution connecting W'_1 to W'_2 . $\Theta_1(\gamma, \delta, W_1, W_2)$ is then defined as the acute angle between the line connecting W_2 to W'_3 and the vertical axis passing through W_2 (cf. Fig. 3). As for $\Theta_2(\gamma, \delta, Z_1, Z_2)$, similarly, we first solve the Riemann problem connecting $Z'_1 = e^{-a\delta}Z_1$ and $Z'_2 = e^{-a\delta}Z_2$, and translate the composite wave curve together with the line connecting Z'_1 to Z'_2 , until it lies over the line connecting Z_1 to Z_2 . But now, instead of considering the translated Riemann solution and its middle state, we consider the composite formed by the translation of the 2-shock curve ending in Z'_2 , so that the translated curve starts now at Z_1 , and the translation of the 1-wave curve, say, 1-shock curve, so that the corresponding translated curve ends up at Z'_2 (cf. Fig. 2). Then we consider the middle state of the composite curve obtained after these translations and call it Z''_3 . Θ_2 is defined as the angle formed between the line connecting Z_2 to Z''_3 and the horizontal axis passing through Z_2 . We complete the definition of both Θ_1 and Θ_2 for $\delta = 0$, by defining $\Theta_1(\gamma, 0, W_1, W_2)$ as the angle that the tangent to 1-shock curve connecting W_1 to W_2 , at W_2 , forms with the vertical axis through W_2 . Similarly, $\Theta_2(\gamma, 0, Z_1, Z_2)$ is the angle that the tangent to the 2-shock curve connecting Z_1 to Z_2 , at Z_2 , forms with the horizontal axis through Z_2 .

For $\gamma = 1$, we have $\Theta_1(1, \delta, W_1, W_2) > \pi/4$, for any $W_1 \in K$, $W_2 \in \bar{V}$ connected by a 1-shock to W_1 , and $0 \leq \delta \leq 1$. Assuming the continuity of Θ_1 with respect to $(\gamma, \delta, W_1, W_2)$, we deduce that $\Theta_1(\gamma, \delta, W_1, W_2)$ for $|\gamma - 1|$ sufficiently small, depending on K and \bar{V} . Similarly, assuming the continuity of Θ_2 with respect to $(\gamma, \delta, Z_1, Z_2)$, we obtain that $\Theta_2(\gamma, \delta, Z_1, Z_2) > \pi/4$, for $|\gamma - 1|$ sufficiently small.

Now, as we have seen in the case $\gamma = 1$, the fact that $\Theta_1 > \pi/4$ and $\Theta_2 > \pi/4$ implies the condition **(D3)**, and this finishes the verification of conditions **(D1)**–**(D4)** for $|\gamma - 1| > 0$ sufficiently small. It would remain only to prove the continuity of Θ_1 and Θ_2 with respect to their arguments, which, although intuitive, requires some topological digression and we leave the details to be given elsewhere.

4.1. Extension to $|a - b| > 0$. We now outline how one can extend the verification of **(D1)**–**(D4)** for $|a - b| > 0$, sufficiently small. We keep considering the systems (3.1), (3.5) and (3.9), for $|\gamma - 1| > 0$. We first see that the conditions **(D1)** and **(D2)** do not require any checking concerning whether $a = b$ or $a \neq b$. So, only conditions **(D3)** and **(D4)** need to be discussed concerning the extension of the application of Theorem 1.1 to the gas dynamics systems with a damping term according to the prescription in **(D2)** when $|a - b| > 0$. Let us set $\mu := b - a$. We again assume K to be a given compact in the physical domain in the Riemann invariants plane and V to be an open neighborhood of K whose closure is a compact in the physical domain.

In order to extend **(D3)** to the case where $|a - b| > 0$, we consider again the functions Θ_1 and Θ_2 introduced above, but now we consider them also depending on μ , thus $\Theta_1 = \Theta_1(\gamma, \delta, \mu, W_1, W_2)$, $\Theta_2 = \Theta_2(\gamma, \delta, \mu, Z_1, Z_2)$, so that $\Theta_1(\gamma, \delta, W_1, W_2)$ and $\Theta_2(\gamma, \delta, Z_1, Z_2)$ as defined before now correspond to the values of these functions when $\mu = 0$. The definition of these functions for $\mu \neq 0$ is similar as that for $\mu = 0$, but now, in the case of Θ_1 , the line connecting $W'_1 = e^{-a\delta}W_1$ and $W'_2 = e^{-b\delta}W_2$ is no longer parallel to that connecting Z_1 and Z_2 , which cause no significant difference. We again translate the line connecting W'_1 to W'_2 , together with the composite wave curve representing the Riemann solution between W'_1 and W'_2 , until W'_1 goes over unto W_1 . Again Θ_1 is defined as the acute angle that the line connecting W_2 to the translated middle state of the Riemann solution W'_3 makes with the vertical axis passing through W_2 . Similarly, we extend the definition of Θ_2 from the case $\mu = 0$ to the case where $\mu \neq 0$.

Again we assume continuity of Θ_1 and Θ_2 with respect to $(\gamma, \delta, \mu, W_1, W_2)$ and $(\gamma, \delta, \mu, Z_1, Z_2)$, respectively. Now, we have $\Theta_1(1, \delta, 0, W_1, W_2) > \pi/4$, for all $\delta \in [0, 1]$, $W_1 \in K$ and $W_2 \in \bar{V}$ belonging to the right 1-shock curve issuing from W_1 . Also, similarly, $\Theta_2(1, \delta, 0, Z_1, Z_2) > \pi/4$, for all $\delta \in [0, 1]$, $Z_1 \in K$ and $Z_2 \in \bar{V}$ belonging to the right 2-shock curve issuing from Z_1 . Therefore, continuity allows us to obtain $\varepsilon_0 > 0$ sufficiently small such that, for $|\gamma - 1| < \varepsilon_0$ and $|\mu| < \varepsilon_0$, we still have $\Theta_1(\gamma, \delta, \mu, W_1, W_2) > \pi/4$ and $\Theta_2(\gamma, \delta, \mu, Z_1, Z_2) > \pi/4$, and this implies the verification of **(D3)**.

As for **(D4)**, let us consider the strictly convex entropies corresponding to each of the systems (3.1), (3.5) and (3.9) which are strictly dissipative when $a = b$. We

write

$$\begin{aligned}\nabla_U \eta_*(U) \cdot G(U) &= a(\omega \partial_\omega \eta_*(U(\omega, \zeta)) + \zeta \partial_\zeta \eta_*(U(\omega, \zeta))) + (b - a) \zeta \partial_\zeta \eta_*(U(\omega, \zeta)) \\ &= A + B.\end{aligned}$$

Now, by the strict convexity of η_* , and $\eta_*(\bar{U}) = 0$, $\nabla \eta_*(\bar{U}) = 0$, we see that for $|b - a| < \varepsilon_0$, with ε_0 as in the preceding paragraph, there exists a sufficiently small neighborhood of \bar{U} , say, for $|U - \bar{U}| < r$, for $r > 0$ sufficiently small, such that $|B| \leq A$ for $|U - \bar{U}| < r$, and so $A + B \geq 0$ in this neighborhood. On the other hand, for $|U - \bar{U}| \geq r$, with U belonging to the compact corresponding to K , $\tilde{K} = (\omega, \zeta)^{-1}(K)$, since $A > 0$ on $\tilde{K} \cap \{|U - \bar{U}| \geq r\}$, we see that for $|b - a|$ sufficiently small $A > |B|$ on $\tilde{K} \cap \{|U - \bar{U}| \geq r\}$. Therefore, we may find $0 < \varepsilon_1 \leq \varepsilon_0$ such that for $|b - a| < \varepsilon_1$ condition **(D4)** is satisfied.

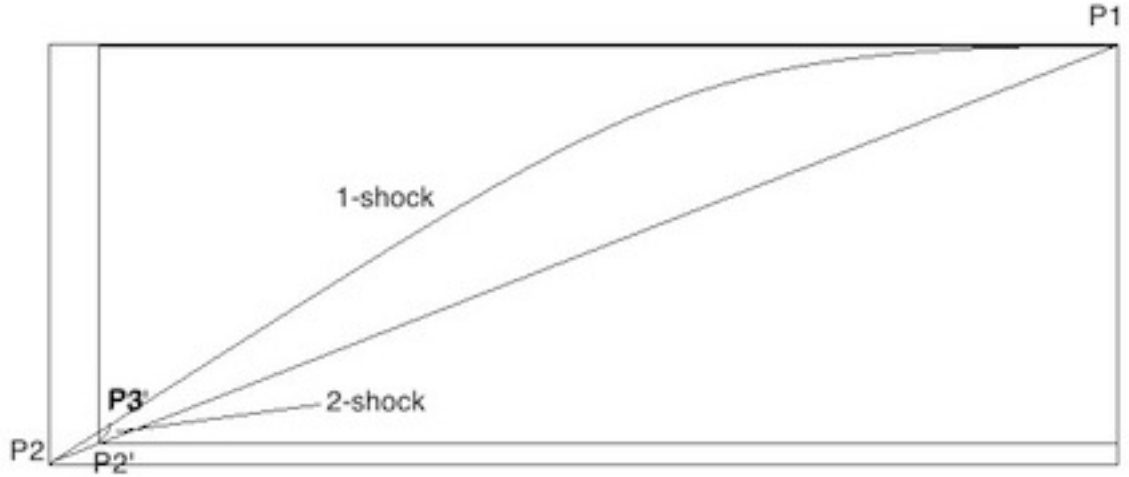
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FIGURE 1. A 1-shock before and after the second part of the fractional step.



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FIGURE 2. A 2-shock before and after the second part of the fractional step.

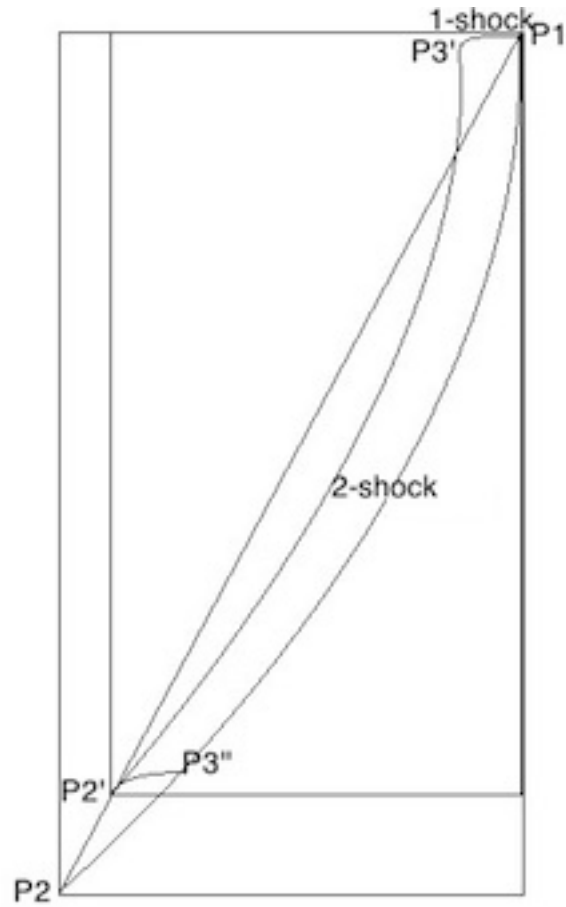


FIGURE 3. The definition of the function Θ_1 .